

# Diffusion and entropy production for multi-networks with fitness factors

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A diffusion process at a special class of multi-networks consisting of weakly coupled networks is analytically solved by an appropriate separation of time scales and by reducing the system dynamics to a Markov chain for aggregated variables. A presence of fitness factors describing attractiveness of individual nodes is taken into account. In the case of system of two coupled networks an equation analogous to the First Fick Law with an additional driving force and a corresponding diffusion constant are found. The entropy production is a sum of entropy changes resulting from a network heterogeneity and the entropy of the Markov chain. Our approach can be also used for hierarchical networks where several different time scales are present.

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(to be modified...). Many systems can be described as structures of special units that are linked together in a form of networks. If these units are not compact objects but possess their internal network structure then one can use a framework of networks of networks or multiplex networks for description of their properties. Starting from Buldyrev et al. [10] coupled complex networks are intensively investigated. There were considered their robustness [12], cascades of failures [10, 11], problem of cooperation [8] and some percolation properties [13]. Authors of those papers were introducing the idea of networks of networks. The idea of networks of networks was successful because the real phenomenon (sociological or technical) seems to be more similar to a few independent networks connected with each other than to one huge network. This approach allows to account not only the global parameters but also the local impact between networks. The simplest example of networks of networks are two connected independent networks. This is also the most important case, because each network of networks consists of some pairs of networks.

The diffusion processes are the objects of interest for researchers from different fields. They are important in the experimental and theoretical physics [2], as well as in the information theory [3]. Recently, the diffusion on complex networks [4–7] was investigated using the theory of Markov chains [9].

Diffusion dynamics in a system of multiplex networks where nodes are conserved through the different layers of the multiplex have been considered very recently in the [1]. It has been shown that the Laplacian matrix of the multiplex network can be constructed from corresponding matrices of each layer. As result one can estimate the effective diffusion in the multiplex due to the layer-layer coupling. It was shown that a diffusion time scale in the multiplex is always shorter than a corresponding time scale calculated for the slower independent network due to the acceleration by the faster partner. The

striking result is that for a strong interlayer coupling a superdiffusion effect can occur, i. e. the effective diffusion in multiplex is faster than in both separate networks.

Here we consider a diffusion process on graph consisting of weakly connected complex networks. Each node of connected networks possesses its attractiveness, and particles choose a new node with a probability proportional to this parameter. There is no interaction between particles. Because of weak internetwork coupling one can separate a fast diffusion processes inside every network from a slower diffusion between different networks and find an approximated analytical solution for this problem. Our system is different from the model considered in [1] where a group of nodes was belonging to both networks that made possible to develop an effective analytical approach based on spectral properties of a corresponding adjacency matrix.

Let  $N$  particles randomly diffuse on an undirected unweighed network  $\mathcal{A}$  with an adjacency matrix  $A \in \mathbb{M}^{M \times M}(\{0, 1\})$  where every node  $i$  ( $i = 1, 2, \dots, M$ ) possesses its special fitness (attractiveness)  $f_i > 0$  that defines its attraction strength for the moving particles. Evolution of particles density  $n_i(t)$  at node  $i$  is given by

$$n_i(t+1) = f_i \sum_{k=1}^M \frac{A_{ik}}{g_k} n_k(t), \quad (1)$$

where  $g_k = \sum_l A_{lk} f_l$  is a total attractiveness of node  $k$  neighbourhood. Let us stress that the fitness parameters defined by the vector  $\mathbf{f} \in \mathbb{R}_+^M$  are not the same as considered in [22] where they determine an evolution of network topology that is time-independent in our case. The equilibrium density of particles follows from an ergodic distribution  $\mu_i$  of a corresponding Markov chain [9]

$$\mu_i = \frac{f_i g_i}{\sum_{k=1}^M f_k g_k}, \quad (2)$$

as  $n_i = N \mu_i$ .

The Shannon entropy (per particle) of the equilibrium distribution can be calculated as  $S_\infty = \sum_{k=1}^M \mu_k \log \mu_k$ . A short algebra leads to

$$S_\infty = -\frac{(f_i g_i \log(f_i g_i))_{\text{av}}}{(f_i g_i)_{\text{av}}} + \log(f_i g_i)_{\text{av}} + \log M, \quad (3)$$

where  $(\cdot)_{\text{av}}$  denotes averaging over all nodes of graph. The term  $\log M$  at rhs of (3) describes the uniform distribution of particles  $\mu_i = 1/M$  that corresponds to the maximal entropy. It is easy to prove that  $S_\infty$  is always less than  $\log M$ , which is a simple consequence of convexity of function  $f(x) = x \log x$  and Jensen's inequality [3]. The non-positive valued sum of the first two terms at the rhs (3)  $s_\infty = S_\infty - \log M$  describes the lowering of the equilibrium particle entropy induced by the complexity of network topology. It equals to zero when every node possesses the same fitness factors  $f_i = f$  and the network is a regular graph, where  $k_i = k_{\text{av}}$ .

Let the network  $\mathcal{A}$  can be represented as a pair of two weakly connected (sub-)networks  $\mathcal{A}^{(1)}$  and  $\mathcal{A}^{(2)}$  (see Fig. 1 (a)) of sizes  $M^{(1)}$  and  $M^{(2)}$  respectively (further we shall consider a larger number of sub-networks). Internal connections between nodes belonging to the same network  $a$  are described by the adjacency matrices  $A^{(a)} \in \mathbb{M}^{M^{(a)} \times M^{(a)}}(\{0, 1\})$ , while the matrix  $C \in \mathbb{M}^{M^{(1)} \times M^{(2)}}$  corresponds to interlinks connecting both networks. The global adjacency matrix is

$$A_{\text{total}} = \begin{bmatrix} A^{(1)} & C \\ C^T & A^{(2)} \end{bmatrix}. \quad (4)$$

We assume only weak coupling between both networks and  $C$  is a sparse random matrix with a parameter  $p_c \ll 1$  describing a probability that a pair of nodes belonging to two different networks  $\mathcal{A}^{(1)}$  and  $\mathcal{A}^{(2)}$  is directly connected. The global vector of attractiveness  $\mathbf{f} \in \mathbb{R}_+^{M^{(1)}+M^{(2)}}$  consists of corresponding vector components  $\mathbf{f}^{(a)} \in \mathbb{R}_+^{M^{(a)}}$  describing attractiveness of nodes in separated networks. The vector of neighbourhood  $\mathbf{g} \in \mathbb{R}_+^{M^{(1)}+M^{(2)}}$  should be calculated by taking into account also interlinks introduced by the matrix  $C$  however because of weak internetwork coupling its components can be approximated by neighbourhood attractiveness of separated networks  $\mathbf{g}^{(a)} \in \mathbb{R}_+^{M^{(a)}}$   $a = 1, 2$ .

Let us assume that a flow between a node  $i$  belonging to a network  $\mathcal{A}^{(a)}$  to all nodes in this network is much larger than a flow from this node to nodes in the network  $\mathcal{A}^{(b)}$  where  $a, b = 1, 2$  and  $a \neq b$ . Such a situation takes place when an attractiveness of a node neighbourhood in the network  $a$  is much larger than an attractiveness coming from nodes belonging to the network  $b$  and connected to this node

$$g_i^{(a)} \gg p_c M^{(b)} f_{\text{av}}^{(b)}, \quad i = 1, 2, \dots, M^{(a)}, \quad (5)$$

The condition above is fulfilled since we have assumed  $p_c \ll 1$ . When flows inside the networks are much larger

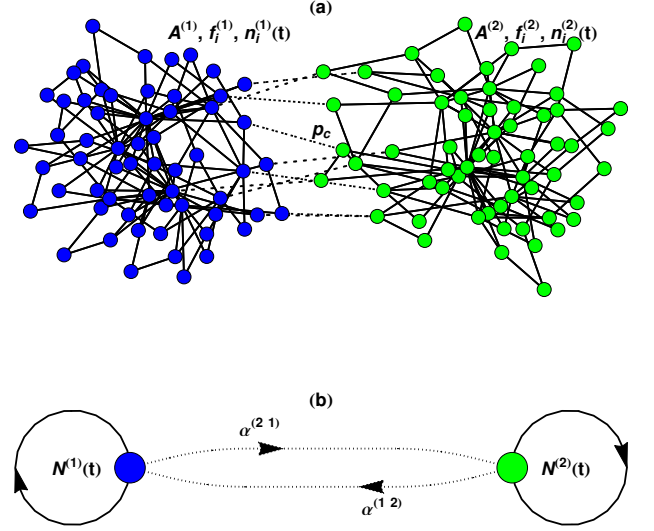


FIG. 1. Connected complex networks. Figure (a) presents two connected Barabási-Albert networks (each consists of 60 nodes). Figure (b) illustrates separating time scales approximation, which can be described as looking at two connected networks as one two-state graph. Diffusion in every graph is faster than diffusion between graphs.

than between them then characteristic time scales for internal processes are much shorter than for processes taking place between the networks. There is not only physical reason for the separation of time scales under the assumption (5). According to the paper of Lawler and Sokal [23] about Cheeger inequality for the Markov Chains the second eigenvalue of the Markov operator is bounded by the constant which is connected to the slowest transition from one subset of graph to its complement. In our case, assumptions (5) provide boundary to the time of reach the equilibrium on the system. Because of the weak connection equilibrium on every networks will be reached faster than equilibrium between them, which implies separation of time scales.

Suppose that fitness of every node is constant and equal to  $f > 0$  and there are no degree-degree correlations in any network. An average attractiveness of the neighbourhood in a network  $a$  is then equal to  $(g^{(a)})_{\text{av}} = f k_{\text{av}}^{(a)}$ ,  $a = 1, 2$ . The relations (5) leads to  $k_{\text{av}}^{(a)} \gg M^{(b)} p_c$ ,  $a, b = 1, 2$ ,  $a \neq b$ . It means an average inter-network degree for  $a$  must be smaller than an internal degree. It is interesting to compare a validity of the mentioned assumption in cases when connected networks are two Erdős-Rényi graphs and when they are two Barabási-Albert networks. Let assume that in the first case graphs possess equal number of nodes and are described by parameters  $p^{(1)}$  and  $p^{(2)}$  corresponding to probabilities that two randomly chosen nodes are directly connected in a

given graph. Then we get  $p^{(1)}, p^{(2)} \gg p_c$  and clearly these conditions do not depend on size of networks so for every size of networks we can use the same  $p_c$ . This relation was received when we considered an average degree in Erdős-Rényi graph what is justified since a dispersion of the degree distribution is negligible. In Barabási-Albert networks case (described with parameters  $m^{(1)}$  and  $m^{(2)}$  respectively) the situation is different since a degree distribution is a power law. Taking a necessary condition for the minimal node degree we receive  $m^{(1)}, m^{(2)} \gg Mp_c$ . In that case an appropriate value of  $p_c$  is related to network sizes.

Let  $n_i^{(a)}(t)$  where  $i = 1, 2, \dots, M^{(a)}$ ,  $a = 1, 2$ , describes the number of particles in the  $i$ -th node of the  $a$ -th network. We assume

$$n_i^{(a)}(t) = N^{(a)}(t)\mu_i^{(a)} \quad (6)$$

where  $N^{(a)}(t)$  are the total number of particles in the  $a$ -th network at time  $t$  and  $\mu_i^{(a)}$  are equilibrium distributions of density of particles in the non-connected networks. These distributions are defined by Eq. (2). The sum of numbers of all particles is preserved  $N^{(1)}(t) + N^{(2)}(t) = N = \text{const}$ .

Using the above approximation we get from Eq. (1)

$$N^{(1)}(t+1) = (1 - \alpha^{(21)})N^{(1)}(t) + \alpha^{(12)}N^{(2)}(t), \quad (7)$$

where  $\alpha^{(ab)}$  are constants dependent on  $A^{(1)}$ ,  $A^{(2)}$ ,  $p_c$ ,  $\mathbf{f}^{(1)}$ ,  $\mathbf{f}^{(2)}$  as follows

$$\alpha^{(ba)} = p_c M^{(b)} \frac{f_{av}^{(a)} f_{av}^{(b)}}{(f_i^{(a)} g_i^{(a)})_{av}}, \quad (8)$$

where  $a, b = 1, 2$  and  $a \neq b$  and a corresponding equation for  $N^{(2)}(t+1)$  follows directly from Eq. 7 by appropriate symmetry relations. The parameters  $\alpha^{(12)}$  and  $\alpha^{(21)}$  describe integrated transition probabilities between the networks at the macroscopic level (see Fig. 1) and can be further simplified when there is no correlation between a node degree and a node fitness factor  $\mathbf{f}^{(a)}$  e.g.  $(f_i^{(a)} g_i^{(a)})_{av} = f_{av}^{(a)} g_{av}^{(a)}$ . Then

$$\alpha^{(ba)} = \frac{f_{av}^{(b)} p_c M^{(b)}}{f_{av}^{(a)} k_{av}^{(a)}}, \quad (9)$$

where  $a, b = 1, 2$ ,  $a \neq b$  and  $\alpha^{(12)}, \alpha^{(21)} \ll 1$  from Eq. (5).

Using initial conditions  $N^{(1)}(0) = (1/2 + \varepsilon)N$  and  $N^{(2)}(0) = (1/2 - \varepsilon)N$  one gets [3]

$$N^{(1)}(t) = C_1 N + C_2^t (1/2 + \varepsilon - C_1) N. \quad (10)$$

where  $C_1 = \alpha^{(21)} / (\alpha^{(12)} + \alpha^{(21)})$  and  $C_2 = 1 - \alpha^{(12)} - \alpha^{(21)}$ .

It follows there is an equilibrium number of particles in the  $\mathcal{A}^{(1)}$  network  $N_\infty^{(1)} = \alpha^{(21)}N / (\alpha^{(12)} + \alpha^{(21)})$ . Near

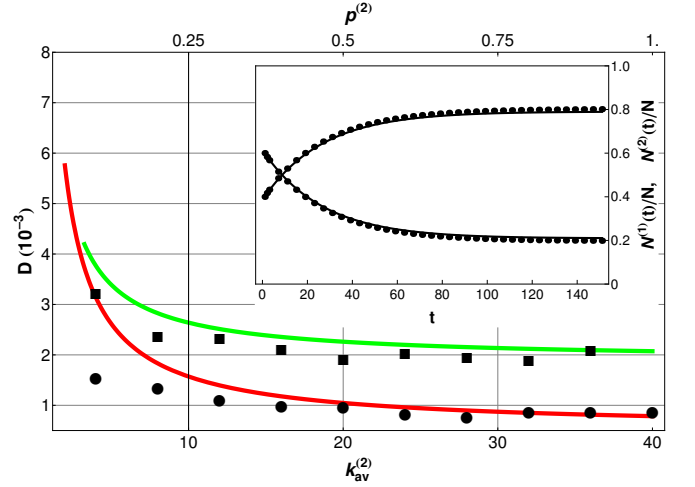


FIG. 2. Influence of internal network density on a diffusion constant for two weakly connected complex networks. The red line (analytical solution) and circles (numerical simulations) correspond to a pair of Barabási-Albert networks. Each of them consists of  $M = 500$  nodes, the mean internal degree of the  $\mathcal{A}^{(1)}$  network is constant and equal to  $k_{av}^{(1)} = 20$  while the corresponding parameter  $k_{av}^{(2)}$  of the  $\mathcal{A}^{(2)}$  network is changing (see the bottom frame). Probability of internetwork connections  $p_c = 4 \cdot 10^{-5}$ . The green line (analytical solution) and squares (numerical simulations) correspond to a pair of Erdős-Rényi graphs. Each of them consists of  $M = 200$  nodes, the mean internal degree  $k_{av}^{(1)}$  of the  $\mathcal{A}^{(1)}$  network is constant and it equals to  $Mp^{(1)} = 20$  where  $p^{(1)} = 0.1$  is a probability for an intra-network link existence between two randomly chosen nodes in the network. A corresponding parameter  $p_2$  of the  $\mathcal{A}^{(2)}$  network (hence its intra-network density  $k_{av}^{(2)} = Mp^{(2)}$ ) is changing (see the top frame). Probability of internetwork connections  $p_c = 3.75 \cdot 10^{-5}$ . The inset shows an example of numerical evolution of total numbers of particles in both networks used for fitting and  $\alpha^{(1)}$  and  $\alpha^{(2)}$  characteristic parameters (see text). In each case dynamics is non-preferential, i.e.  $f_{i_1}^{(1)} = f_{i_2}^{(2)} = \text{const}$ ,  $i_a = 1, \dots, M^{(a)}$ ,  $a = 1, 2$ .

the equilibrium state variations of  $N^{(1)}(t)$  are sufficiently slow and then it is possible to write Eq. (7) in a form analogous of the First Fick's Law [21] with a diffusion constant  $D$  and in the presence of an additional internetwork driving force  $F$

$$\dot{N}^{(1)}(t) = - \left( N^{(1)}(t) - N^{(2)}(t) \right) D + FN, \quad (11)$$

where

$$D = \frac{\alpha^{(12)} + \alpha^{(21)}}{2}, \quad F = \frac{\alpha^{(21)} - \alpha^{(12)}}{2}. \quad (12)$$

Let us note that the driving force  $F$  can cause a non-zero flux  $\dot{N}^{(1)}(t)$  even if there is no gradient  $N^{(1)}(t) - N^{(2)}(t)$ . The force  $F$  results from differences in the network sizes  $M^{(a)}$ , from differences in the network mean fitness  $\mathbf{f}^{(a)}$  and from the differences in the mean network degrees  $k_{av}^{(a)}$  because of the neighbourhood attractiveness

$\mathbf{g}^{(a)}$ ,  $a = 1, 2$ . The last dependence can be interpreted as a topological influence on the diffusion process, e.g. both networks can be attributed some potentials dependent on their topology. It follows changes in networks internal topology can induce a particle flux between the networks, e.g. adding a new link between two nodes in the  $\mathcal{A}^{(1)}$  network increases its mean degree  $k_{av}^{(1)}$  thus the parameter  $\alpha^{(12)}$  decreases and more particles are transported to this network. The denser is the network the larger number of particles  $N_\infty^{(j)}$  will it attract in the equilibrium state. It is interesting from the applicational point of view - by changing structures of the networks or their connections one can modify not only diffusion constants but also change a direction of particles flow.

The above calculations show a connection between macroscopic quantities, i.e. the diffusion constant  $D$  or the driving force  $F$  and internal or internetwork topological properties. Fig. 2 presents an influence of internal networks link density on a diffusion constant for two connected Barabási-Albert networks and two connected Erdős-Rényi graphs. Analytical calculations following from Eqs. (8), (12) well correspond to numerical results obtained from random walks simulation. Parameters  $\alpha^{(12)}$  and  $\alpha^{(21)}$  and the resulting diffusion constant  $D$  were received from the best fit Eq. (10) to numerically received dependences  $N^{(1)}(t)$  and  $N^{(2)}(t)$  - see the figure inset. As we can see on graph - the diffusion constant  $D$  is decreasing with increasing an internal density of a network. It is understandable - when a network becoming denser, its easier keeps particles in its interior since a diffusing particle more frequently selects an internal links as compared to external ones. It is reflected by decrease of parameters  $\alpha^{(12)}$  and  $\alpha^{(21)}$  and hence the diffusion constant  $D$  as functions of internal degrees  $k_{av}^{(a)}$ , see Eqs. 9.

Using the approximation of separation of the time scales (6) the total entropy of the system can now be written as

$$S_{\text{total}}(t) = \left[ N^{(1)}(t)S_\infty^{(1)} + N^{(2)}(t)S_\infty^{(2)} \right] + \left[ -N^{(1)}(t) \log N^{(1)}(t) - N^{(2)}(t) \log N^{(2)}(t) + N \log N \right], \quad (13)$$

where  $S_\infty^{(a)}$ ,  $a = 1, 2$ , are equilibrium entropies per particle given by Eq. (3) for the  $\mathcal{A}^{(1)}$  and the  $\mathcal{A}^{(2)}$  network. The first square bracket in (13) results from lack of information about particle position inside the given network. The second bracket in (13) is the entropy of the two-state Markov chain (see Fig. 1 (b)) and results from a lack of information about a particles sharing between both networks.

Now the entropy changes can be written by taking the

time derivative of the  $S_{\text{total}}$  given by (13)

$$\sigma(t) = \dot{N}^{(1)}(t) \left( \log N^{(2)}(t) - \log N^{(1)}(t) + S_\infty^{(1)} - S_\infty^{(2)} \right), \quad (14)$$

where  $S_\infty^{(a)}$ ,  $a = 1, 2$  are the equilibrium entropy per particle in each network (see Eq. (3)). It follows changes of system entropy come from entropy at the macroscopic level (two-state Markov chain, see Fig. 1) and transport of microscopical entropies related to network topologies  $S_\infty^{(a)}$ , for  $a = 1, 2$  that can be different in both networks.

Up to now we have considered the model of two coupled networks. It is easy to show that our approach is valid also for any system of  $m$  weakly coupled networks i.e. for networks of networks or for multiply networks. Then instead of equation (7) one gets

$$N^{(a)}(t+1) = \sum_{l=1}^m \alpha^{(al)} N^{(l)}(t), \quad (15)$$

where  $\alpha^{(ab)}$  describes a strength of flow from a network  $\mathcal{A}^{(b)}$  to  $\mathcal{A}^{(a)}$  for  $a \neq b$ .

$$\alpha^{(ab)} = p_c^{(ab)} M^{(a)} \frac{f_{av}^{(a)} f_{av}^{(b)}}{(f_i^{(b)} g_i^{(b)})_{av}}, \quad \alpha^{(aa)} = 1 - \sum_{r \neq a} \alpha^{(ra)}. \quad (16)$$

Then equation (11) changes to

$$\dot{N}^{(a)}(t) = \sum_{l \neq a} [-D^{(la)}(N^{(a)}(t) - N^{(l)}(t)) + F^{(la)}(N^{(a)}(t) + N^{(l)}(t))], \quad (17)$$

$$+ F^{(la)}(N^{(a)}(t) + N^{(l)}(t))], \quad (18)$$

where

$$D^{(ab)} = \frac{\alpha^{(ab)} + \alpha^{(ba)}}{2}, \quad F^{(ab)} = \frac{\alpha^{(ba)} - \alpha^{(ab)}}{2}. \quad (19)$$

The equilibrium distribution of  $N^{(a)}$  follows from a fixed point of the linear operator at the rhs (15) and its second eigenvalue describes the rate this equilibrium is approached. For the case of multiply networks the production of entropy (14) is

$$\sigma(t) = \sum_{r=1}^m \dot{N}^{(r)}(t) [S_\infty^{(r)} - \log N^{(r)}(t)]. \quad (20)$$

The approach is valid also if one considers diffusion on weighted networks with self-loops. Then the symmetrical adjacency matrix  $A$  in Eq. (1) is defined by non-negative elements including diagonal elements. Looking at (15) one finds that it possesses the structure of (7) when parameters  $\hat{f}_a$ ,  $\hat{g}_a$  and  $\hat{A}_{ab}$ ,  $a, b = 1, \dots, m$ , for higher level ( $\hat{X}$  means parameter  $X$  for the higher level) are appropriately defined, e.g.

$$\hat{f}_a = M^{(a)} f_{av}^{(a)}, \quad \hat{A}_{ab} = p_s^{(ab)}, \quad (21)$$

It follows one can consider iterations of our approach for a next level when groups of networks are weakly connected to the group of networks. Thus one is in possession to consider a system of hierarchically build weakly-coupled networks.

Conclusions (to be extended...).

In this paper we have considered diffusion processes on weakly connected complex networks with a special fitness parameters of individual nodes. We have introduced an analytical approach that bases on separation of time scales describing dynamics at different levels. The approximation makes possible an analytical estimation of equilibrium distribution of diffusing particles and a production of entropy in multinetworks as well as in hierarchical systems of weakly coupled networks.

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